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NAVAL POSTGRADUATE SCHOOL

Monterey, California

ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

by

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ABSTRACT:

A complete asymptotic development of the Stirling numbers S(N,K) of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans. Roy. Soc. Canad., 49(1955) 49-54] and de Bruijn [Asymptotic Methods in Analysis, North-Holland Publishing Co., Amsterdam, (1958) 102-109] for the asymptotic representation of the related Bell numbers.

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ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

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A complete asymptotic development of the Stirling numbers $S(N,K)$ of the second kind is obtained by the saddle point method.

1. INTRODUCTION

Hsu [1] has given the asymptotic expansion

$$S(N,K) \sim (\tfrac{1}{2}K^2)^{N-K}[1 + \sum_{s=1}^{t} K^{-s} f_s(N-K) + O(K^{-t-1})]/(N-K)! \qquad (1)$$

for Stirling numbers $S(N,K)$ of the second kind, where $f_s(N-K)$ are polynomials of argument $N-K$ and $f_s(0) = 0$. The expansion (1) is useful only for N-K small, as is indicated in Section 3. We obtain a complete asymptotic expansion of $S(N,K)$ in powers of $(N+1)^{-1}$ valid for all K, using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers.

2. ASYMPTOTICS OF S(N,K)

A generating function for $S(N,K)$ is

$$\left(\frac{e^z-1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N,K)\, z^{N-K} . \qquad (2)$$

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Hence, the Cauchy integral formula gives

$$S(N,K) = \frac{N!}{2\pi i K!} \int_C (e^z-1)^K z^{-N-1} dz \tag{3}$$

where the contour C encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$(t-z)e^{z-t} = te^{-t} , \tag{4}$$

where $t = (N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z = u$ is on the positive real axis with $t - 1 < u < t$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z = u$. Since there are no other roots of (4) for $|t - z| \leq t - u$, we may apply the Lagrange inversion formula to obtain

$$u = t - \sum_{m=1}^{\infty} m^{m-1}(te^{-t})^m/m! \tag{5}$$

convergent for $t > 1$. Another form of (4) is the identity

$$K = (N+1)(1-e^{-u})/u \tag{6}$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour C descending from $z = u$ is taken as the line $z = u + iy$, $-\infty < y < \infty$, parallel to the imaginary axis. This path has the property that the modulus of the integrand in (3) is maximal at the saddle point $z = u$. The closed contour C is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$i(e^u-1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u+iy)dy \tag{7}$$

where

$$\psi(z) = K \ln[(e^z-1)/(e^u-1)] - (N+1) \ln(z/u) . \tag{8}$$

The contribution of the various parts of the $z = u + iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \mathrm{Re}\, \psi(z)$ we have to study

$$\mathrm{Re}\, \psi(u+iy) = K \ln[(e^{2u}-2e^u\cos y+1)^{\frac{1}{2}}/(e^u-1)]-(N+1)\ln(1+y^2u^{-2})^{\frac{1}{2}} .$$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since

$$1 + y^2u^{-2} \geq 1 + \pi(2y-\pi)u^{-2}$$

for $y \geq \pi$ we have

$$(e^u-1)^K u^{-N-1} \left| \int_{\pi}^{\infty} \exp \psi(u+iy)\, dy \right| < u^{1-N}(e^u+1)^K/\pi(N-1)(1+\pi^2u^{-2})^{\frac{1}{2}N-\frac{1}{2}}$$

which is of $O(N^{-N}e^N)$ for K small and of $O(N^{-1}e^{-0.19N})$ for K large. Since $\psi\mathrm{Re}\, \psi(u+iy)$ is even the part of the integral (7) for

$|y| > \pi$ tends toward zero as $N \to \infty$. We now direct our attention to the interval $|y| < \pi$ where the saddle point at $y = 0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y| < u$, is

$$\psi = -\frac{N+1}{2u}\left(\frac{1}{u} - \frac{1}{e^u-1}\right)y^2 + (N+1)\sum_{j=1}^{\infty} \frac{(iy)^{j+2}}{(j+2)!}\left(\frac{d}{dz}\right)^{j+1}\left[\frac{1-e^{-u}}{u(e^z-1)} - \frac{1}{z}\right]_{z=u} \quad (9)$$

where the identity (6) has been used. We now make the substitutions

$$w = \left[\frac{N+1}{2}\right]^{\frac{1}{2}}\left[1 - \frac{u}{e^u-1}\right]^{\frac{1}{2}} y/u \quad (10)$$

and

$$a_j = (iwu)^{j+2}\left(\frac{d}{dz}\right)^{j+1}\left[\frac{1-e^{-u}}{u(e^z-1)} - \frac{1}{z}\right]_{z=u} \Big/ (j+2)!\left[\tfrac{1}{2} - \frac{\frac{1}{2}u}{e^u-1}\right]^{\frac{1}{2}j+1} \quad (11)$$

to obtain

$$S(N,K) = B\int_{-\infty}^{\infty} e^{-w^2+f[(N+1)^{-\frac{1}{2}}]}dw \quad (12)$$

where

$$B = N!(e^u-1)^K/\pi\sqrt{2(N+1)}K!u^N\sqrt{1+u/(1-\exp u)} \quad (13)$$

and

$$f[(N+1)^{-\frac{1}{2}}] = \sum_{j=1}^{\infty} a_j (N+1)^{-\frac{1}{2}j} \ . \quad (14)$$

On using Cauchy's theorem for derivatives we find that

$$|a_j| < L\,\sigma^j \quad (15)$$

where

$$L = 2w^2u^2/(u-1)(1-e^{1-u})(1 - \frac{u}{e^u-1}) \tag{16}$$

and

$$\sigma = \sqrt{2}wu/(1 - \frac{u}{e^u-1})^{\frac{1}{2}} . \tag{17}$$

We now expand $\exp f[(N+1)^{-\frac{1}{2}}]$ in a Taylor series of the form

$$\exp f[(N+1)^{-\frac{1}{2}}] = \sum_{j=0}^{\infty} b_j (N+1)^{-\frac{1}{2}j} \tag{18}$$

where $b_0 = 1$ and b_j are even or odd polynomials in w according to j whether it is even or odd. By a lemma of Moser and Wyman [4]

$$|b_j| \leq L\sigma^j (1+L)^{j-1} . \tag{19}$$

By (15) and (19) we see that the series (14) and (18) are convergent for suitably large N. At this point all of the reasoning given by Moser and Wyman [2] for the asymptotic expansion of the Bell numbers applies equally well to the $S(N,K)$ expansion and we conclude that

$$S(N,K) \sim B\{\sum_{j=0}^{s-1} (N+1)^{-j} \int_{-\infty}^{\infty} b_{2j} e^{-w^2} dw + O[(N+1)^{-s}]\} . \tag{20}$$

The first two terms of the expansion (20) have been calculated to be

$$S(N,K) \sim \frac{N!(e^u-1)^K}{\sqrt{2\pi(N+1)}K!u^N\sqrt{1+u/(1-\exp u)}} \times$$

$$\cdot\{1 + \frac{6-u^3(e^{2u}+4e^u+1)(e^u-1)^{-3}}{8(N+1)[1+u/(1-\exp u)]^2} - \frac{5[2-u^2(e^u+1)(e^u-1)^{-2}]^2}{24(N+1)[1+u/(1-\exp u)]^3}\} . \quad (21)$$

3. NUMERICAL EXAMPLE

The following 6-significant-figure table compares the exact values of S(100,K) with values computed from (20) and (1) for k; $1 \leq k \leq 100$:

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
1	1.00000	1.00083	1.00000	$7.74429\ 10^{-176}$
2	$6.33825\ 10^{29}$	$6.34348\ 10^{29}$	$6.33825\ 10^{29}$	$1.81186\ 10^{-115}$
3	$8.58963\ 10^{46}$	$8.59672\ 10^{46}$	$8.58962\ 10^{46}$	$3.59066\ 10^{-80}$
4	$6.69558\ 10^{58}$	$6.70110\ 10^{58}$	$6.69557\ 10^{58}$	$2.98751\ 10^{-55}$
5	$6.57384\ 10^{67}$	$6.57927\ 10^{67}$	$6.57384\ 10^{67}$	$4.46215\ 10^{-36}$
6	$9.07387\ 10^{74}$	$9.08136\ 10^{74}$	$9.07387\ 10^{74}$	$1.41738\ 10^{-20}$
7	$6.41760\ 10^{80}$	$6.42291\ 10^{80}$	$6.41760\ 10^{80}$	$1.23857\ 10^{-7}$
8	$5.05211\ 10^{85}$	$5.05630\ 10^{85}$	$5.05210\ 10^{85}$	$1.38293\ 10^{4}$
9	$7.31910\ 10^{89}$	$7.32522\ 10^{89}$	$7.31910\ 10^{89}$	$5.33906\ 10^{13}$
10	$2.75500\ 10^{93}$	$2.75733\ 10^{93}$	$2.75500\ 10^{93}$	$1.41107\ 10^{22}$
11	$3.44958\ 10^{96}$	$3.45255\ 10^{96}$	$3.44958\ 10^{96}$	$4.16533\ 10^{29}$
12	$1.72552\ 10^{99}$	$1.72704\ 10^{99}$	$1.72552\ 10^{99}$	$1.97468\ 10^{36}$
13	$3.96430\ 10^{101}$	$3.96785\ 10^{101}$	$3.96430\ 10^{101}$	$1.98352\ 10^{42}$
14	$4.66349\ 10^{103}$	$4.66773\ 10^{103}$	$4.66350\ 10^{103}$	$5.24140\ 10^{47}$
15	$3.06221\ 10^{105}$	$3.06502\ 10^{105}$	$3.06221\ 10^{105}$	$4.32888\ 10^{52}$
16	$1.20332\ 10^{107}$	$1.20443\ 10^{107}$	$1.20333\ 10^{107}$	$1.28480\ 10^{57}$
17	$2.99575\ 10^{108}$	$2.99850\ 10^{108}$	$2.99576\ 10^{108}$	$1.53680\ 10^{61}$
18	$4.95343\ 10^{109}$	$4.95793\ 10^{109}$	$4.95345\ 10^{109}$	$8.15008\ 10^{64}$
19	$5.65920\ 10^{110}$	$5.66423\ 10^{110}$	$5.65922\ 10^{110}$	$2.07645\ 10^{68}$
20	$4.61926\ 10^{111}$	$4.62326\ 10^{111}$	$4.61927\ 10^{111}$	$2.72088\ 10^{71}$

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
21	$2.77170\ 10^{112}$	$2.77402\ 10^{112}$	$2.77171\ 10^{112}$	$1.94406\ 10^{74}$
22	$1.25295\ 10^{113}$	$1.25396\ 10^{113}$	$1.25295\ 10^{113}$	$7.96614\ 10^{76}$
23	$4.35876\ 10^{113}$	$4.36209\ 10^{113}$	$4.35877\ 10^{113}$	$1.95614\ 10^{79}$
24	$1.18873\ 10^{114}$	$1.18959\ 10^{114}$	$1.18873\ 10^{114}$	$2.99138\ 10^{81}$
25	$2.58320\ 10^{114}$	$2.58496\ 10^{114}$	$2.58321\ 10^{114}$	$2.94696\ 10^{83}$
26	$4.53753\ 10^{114}$	$4.54042\ 10^{114}$	$4.53753\ 10^{114}$	$1.92723\ 10^{85}$
27	$6.52519\ 10^{114}$	$6.52905\ 10^{114}$	$6.52520\ 10^{114}$	$8.59300\ 10^{86}$
28	$7.76973\ 10^{114}$	$7.77396\ 10^{114}$	$7.76973\ 10^{114}$	$2.67528\ 10^{88}$
29	$7.73865\ 10^{114}$	$7.74251\ 10^{114}$	$7.73864\ 10^{114}$	$5.94163\ 10^{89}$
30	$6.50629\ 10^{114}$	$6.50922\ 10^{114}$	$6.50628\ 10^{114}$	$9.59702\ 10^{90}$
31	$4.65568\ 10^{114}$	$4.65757\ 10^{114}$	$4.65567\ 10^{114}$	$1.14719\ 10^{92}$
32	$2.85660\ 10^{114}$	$2.85763\ 10^{114}$	$2.85659\ 10^{114}$	$1.03105\ 10^{93}$
33	$1.51311\ 10^{114}$	$1.51358\ 10^{114}$	$1.51310\ 10^{114}$	$7.06841\ 10^{93}$
34	$6.96173\ 10^{113}$	$6.96362\ 10^{113}$	$6.96170\ 10^{113}$	$3.74513\ 10^{94}$
35	$2.79791\ 10^{113}$	$2.79856\ 10^{113}$	$2.79790\ 10^{113}$	$1.55216\ 10^{95}$
36	$9.87317\ 10^{112}$	$9.87504\ 10^{112}$	$9.87311\ 10^{112}$	$5.08767\ 10^{95}$
37	$3.07351\ 10^{112}$	$3.07397\ 10^{112}$	$3.07349\ 10^{112}$	$1.33233\ 10^{96}$
38	$8.47721\ 10^{111}$	$8.47817\ 10^{111}$	$8.47715\ 10^{111}$	$2.81367\ 10^{96}$
39	$2.07993\ 10^{111}$	$2.08009\ 10^{111}$	$2.07991\ 10^{111}$	$4.83323\ 10^{96}$
40	$4.55640\ 10^{110}$	$4.55659\ 10^{110}$	$4.55636\ 10^{110}$	$6.80712\ 10^{96}$
41	$8.94238\ 10^{109}$	$8.94247\ 10^{109}$	$8.94230\ 10^{109}$	$7.91868\ 10^{96}$
42	$1.57729\ 10^{109}$	$1.57725\ 10^{109}$	$1.57727\ 10^{109}$	$7.66078\ 10^{96}$
43	$2.50762\ 10^{108}$	$2.50750\ 10^{108}$	$2.50759\ 10^{108}$	$6.20269\ 10^{96}$
44	$3.60314\ 10^{107}$	$3.60287\ 10^{107}$	$3.60310\ 10^{107}$	$4.22802\ 10^{96}$
45	$4.69092\ 10^{106}$	$4.69045\ 10^{106}$	$4.69087\ 10^{106}$	$2.43965\ 10^{96}$

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
46	$5.54631\ 10^{105}$	$5.54563\ 10^{105}$	$5.54625\ 10^{105}$	$1.19777\ 10^{96}$
47	$5.96839\ 10^{104}$	$5.96753\ 10^{104}$	$5.96832\ 10^{104}$	$5.02740\ 10^{95}$
48	$5.85718\ 10^{103}$	$5.85623\ 10^{103}$	$5.85711\ 10^{103}$	$1.81206\ 10^{95}$
49	$5.25178\ 10^{102}$	$5.25084\ 10^{102}$	$5.25171\ 10^{102}$	$5.63198\ 10^{94}$
50	$4.30983\ 10^{101}$	$4.30900\ 10^{101}$	$4.30977\ 10^{101}$	$1.51529\ 10^{94}$
51	$3.24224\ 10^{100}$	$3.24157\ 10^{100}$	$3.24219\ 10^{100}$	$3.54197\ 10^{93}$
52	$2.23926\ 10^{99}$	$2.23877\ 10^{99}$	$2.23922\ 10^{99}$	$7.21740\ 10^{92}$
53	$1.42177\ 10^{98}$	$1.42146\ 10^{98}$	$1.42175\ 10^{98}$	$1.28610\ 10^{92}$
54	$8.30948\ 10^{96}$	$8.30758\ 10^{96}$	$8.30936\ 10^{96}$	$2.01003\ 10^{91}$
55	$4.47546\ 10^{95}$	$4.47442\ 10^{95}$	$4.47539\ 10^{95}$	$2.76288\ 10^{90}$
56	$2.22373\ 10^{94}$	$2.22322\ 10^{94}$	$2.22370\ 10^{94}$	$3.34862\ 10^{89}$
57	$1.02031\ 10^{93}$	$1.02007\ 10^{93}$	$1.02029\ 10^{93}$	$3.58715\ 10^{88}$
58	$4.32678\ 10^{91}$	$4.32581\ 10^{91}$	$4.32671\ 10^{91}$	$3.40392\ 10^{87}$
59	$1.69718\ 10^{90}$	$1.69682\ 10^{90}$	$1.69716\ 10^{90}$	$2.86713\ 10^{86}$
60	$6.16213\ 10^{88}$	$6.16086\ 10^{88}$	$6.16202\ 10^{88}$	$2.14775\ 10^{85}$
61	$2.07225\ 10^{87}$	$2.07185\ 10^{87}$	$2.07221\ 10^{87}$	$1.43335\ 10^{84}$
62	$6.45799\ 10^{85}$	$6.45686\ 10^{85}$	$6.45787\ 10^{85}$	$8.53612\ 10^{82}$
63	$1.86594\ 10^{84}$	$1.86566\ 10^{84}$	$1.86591\ 10^{84}$	$4.54310\ 10^{81}$
64	$5.00048\ 10^{82}$	$4.99983\ 10^{82}$	$5.00038\ 10^{82}$	$2.16381\ 10^{80}$
65	$1.24327\ 10^{81}$	$1.24315\ 10^{81}$	$1.24325\ 10^{81}$	$9.23409\ 10^{78}$
66	$2.86851\ 10^{79}$	$2.86832\ 10^{79}$	$2.86846\ 10^{79}$	$3.53479\ 10^{77}$
67	$6.14247\ 10^{77}$	$6.14228\ 10^{77}$	$6.14235\ 10^{77}$	$1.21495\ 10^{76}$
68	$1.22080\ 10^{76}$	$1.22082\ 10^{76}$	$1.22078\ 10^{76}$	$3.75283\ 10^{74}$
69	$2.25191\ 10^{74}$	$2.25204\ 10^{74}$	$2.25186\ 10^{74}$	$1.04256\ 10^{73}$
70	$3.85478\ 10^{72}$	$3.85521\ 10^{72}$	$3.85470\ 10^{72}$	$2.60661\ 10^{71}$

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
71	$6.12206\ 10^{70}$	$6.12310\ 10^{70}$	$6.12193\ 10^{70}$	$5.86843\ 10^{69}$
72	$9.01786\ 10^{68}$	$9.01999\ 10^{68}$	$9.01766\ 10^{68}$	$1.19026\ 10^{68}$
73	$1.23149\ 10^{67}$	$1.23187\ 10^{67}$	$1.23147\ 10^{67}$	$2.17566\ 10^{66}$
74	$1.55828\ 10^{65}$	$1.55889\ 10^{65}$	$1.55825\ 10^{65}$	$3.58501\ 10^{64}$
75	$1.82584\ 10^{63}$	$1.82671\ 10^{63}$	$1.82579\ 10^{63}$	$5.32626\ 10^{62}$
76	$1.97939\ 10^{61}$	$1.98054\ 10^{61}$	$1.97934\ 10^{61}$	$7.13564\ 10^{60}$
77	$1.98357\ 10^{59}$	$1.98495\ 10^{59}$	$1.98353\ 10^{59}$	$8.62061\ 10^{58}$
78	$1.83542\ 10^{57}$	$1.83692\ 10^{57}$	$1.83538\ 10^{57}$	$9.39120\ 10^{56}$
79	$1.56619\ 10^{55}$	$1.56769\ 10^{55}$	$1.56615\ 10^{55}$	$9.22442\ 10^{54}$
80	$1.23065\ 10^{53}$	$1.23202\ 10^{53}$	$1.23062\ 10^{53}$	$8.16802\ 10^{52}$
81	$8.88952\ 10^{50}$	$8.90092\ 10^{50}$	$8.88930\ 10^{50}$	$6.51851\ 10^{50}$
82	$5.89155\ 10^{48}$	$5.90025\ 10^{48}$	$5.89140\ 10^{48}$	$4.68689\ 10^{48}$
83	$3.57456\ 10^{46}$	$3.58062\ 10^{46}$	$3.57447\ 10^{46}$	$3.03474\ 10^{46}$
84	$1.98038\ 10^{44}$	$1.98423\ 10^{44}$	$1.98032\ 10^{44}$	$1.76835\ 10^{44}$
85	$9.98901\ 10^{41}$	$1.00113\ 10^{42}$	$9.98874\ 10^{41}$	$9.26435\ 10^{41}$
86	$4.57157\ 10^{39}$	$4.58323\ 10^{39}$	$4.57144\ 10^{39}$	$4.35788\ 10^{39}$
87	$1.89083\ 10^{37}$	$1.89636\ 10^{37}$	$1.89078\ 10^{37}$	$1.83701\ 10^{37}$
88	$7.03510\ 10^{34}$	$7.05874\ 10^{34}$	$7.03487\ 10^{34}$	$6.92073\ 10^{34}$
89	$2.34174\ 10^{32}$	$2.35081\ 10^{32}$	$2.34166\ 10^{32}$	$2.32155\ 10^{32}$
90	$6.92830\ 10^{29}$	$6.95936\ 10^{29}$	$6.92803\ 10^{29}$	$6.89924\ 10^{29}$
91	$1.80767\ 10^{27}$	$1.81711\ 10^{27}$	$1.80759\ 10^{27}$	$1.80434\ 10^{27}$
92	$4.11940\ 10^{24}$	$4.14465\ 10^{24}$	$4.11920\ 10^{24}$	$4.11648\ 10^{24}$
93	$8.10134\ 10^{21}$	$8.16026\ 10^{21}$	$8.10087\ 10^{21}$	$8.09946\ 10^{21}$
94	$1.35401\ 10^{19}$	$1.36586\ 10^{19}$	$1.35391\ 10^{19}$	$1.35393\ 10^{19}$

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
95	$1.88471\ 10^{16}$	$1.90493\ 10^{16}$	$1.88453\ 10^{16}$	$1.88469\ 10^{16}$
96	$2.12499\ 10^{13}$	$2.15369\ 10^{13}$	$2.12469\ 10^{13}$	$2.12499\ 10^{13}$
97	$1.86376\ 10^{10}$	$1.89674\ 10^{10}$	$1.86334\ 10^{10}$	$1.86376\ 10^{10}$
98	$1.19254\ 10^{7}$	$1.22200\ 10^{7}$	$1.19202\ 10^{7}$	$1.19254\ 10^{7}$
99	$4.95000\ 10^{3}$	$5.14199\ 10^{3}$	$4.94451\ 10^{3}$	$4.95000\ 10^{3}$
100	1.00000	1.08086	$9.94331\ 10^{-1}$	1.00000

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13. ABSTRACT

A complete asymptotic development of the Stirling numbers S(N,K) of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans. Roy. Soc. Canad., 49(1955) 49-54] and de Bruijn [Asymptotic Methods in Analysis, North-Holland Publishing Co., Amsterdam, (1958) 102-109] for the asymptotic representation of the related Bell numbers.

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